Which spline spaces for design?

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Abstract

We recently determined the largest class of spaces of sufficient regularity which are suitable for design. We are now interested in connecting different such spaces to obtain the largest class of splines usable for design. *To cite this article: M.-L. Mazure, C. R. Acad. Sci. Paris, Ser. ???? (200?).*

Résumé

Quelles splines utiliser pour le design ? Nous avons récemment déterminé la plus grande classe d'espaces (de fonctions suffisamment régulières) bons pour le design. Comment connecter de tels espaces pour produire de "bons" espaces de splines ? *Pour citer cet article : M.-L. Mazure, C. R. Acad. Sci. Paris, Ser. ? ? ? ? (200 ?).*

1. Introduction

To justify the framework presented in Section 3, we draw the reader's attention on two points. First, a "good" spline space is expected to permit the development of all classical design algorithms, among which any knot insertion algorithm. This is the reason why we naturally have to require the section-spaces themselves to be "good for design". This explains why Section 2 is devoted to a brief presentation of the major results concerning Quasi Extended Chebyshev spaces, see [8] and [9]. Second, the presence of connection matrices in our spline spaces is justified by the fact that requiring that left and right derivatives coincide at a given knot up to some order has no special meaning for parametric spline curves.

2. Quasi Extended Chebyshev spaces

Given a non-trivial real interval I and an integer $n \geqslant 1$, let $\mathbb{E} \subset C^{n-1}(I)$ be an $(n+1)$-dimensional space. It is said to be of *strict dimension* $(n+1)$ if it remains of dimension $(n+1)$ by restriction to any non-trivial subinterval of I. We say that $\mathbb{E}$ is a *Quasi Extended Chebsyhev space* (for short, QEC-space) *on I* when any Hermite interpolation problem in $(n+1)$ data **which is not a Taylor interpolation problem** has a unique solution in $\mathbb{E}$ [8]. By comparison, $\mathbb{E}$ is an $(n+1)$-dimensional EC-space on I when any Hermite interpolation problem in $(n+1)$ data has a unique solution in $\mathbb{E}$. This includes Taylor interpolation problems and therefore being an $(n+1)$-dimensional EC-space on I requires $\mathbb{E}$ to be contained in $C^n(I)$. The interest of QEC-spaces lies in the following result:

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Theorem 2.1 *Given $n \geqslant 2$, let $\mathbb{E} \subset C^{n-1}(I)$ be of strict dimension $(n+1)$ and let it contain constants. The following two properties are then equivalent:*

(i) *$\mathbb{E}$ is "good for design", in the sense that $\mathbb{E}$ possesses blossoms;*

(ii) *the space $D\mathbb{E} := \{DF := F' \mid F \in \mathbb{E}\}$ is an n-dimensional QEC-space on I.*

The reader interested in the exact definition of blossoms can refer to [8]. Let us just recall that these powerful tools are defined in a geometrical way using intersection of osculating flats. Our terminology *"good for design"* is highly justified by the fact that, when (i) is satisfied, each function $F \in \mathbb{E}$ *"blossoms"* into a function $f : I^n \to \mathbb{R}$, called *the blossom of F* which satisfies the following properties:

$(B)_1$ symmetry: f is symmetric on I^n;

$(B)_2$ diagonal property: for all $x \in I$, $f(x^{[n]}) = F(x)$, $x^{[n]}$ standing for x repeated n times;

$(B)_3$ pseudoaffinity property: given any $y_1, \dots, y_{n-1}, a, b \in I$, with $a < b$, there exists a strictly increasing function $\beta(y_1, \dots, y_{n-1}; a, b; \boldsymbol{.}) : I \to \mathbb{R}$ (independent of F) such that:

$$\begin{aligned} f(y_1, \dots, y_{n-1}, x) = \big[1 - \beta(y_1, \dots, y_{n-1}; a, b; x)\big] f(y_1, \dots, y_{n-1}, a) \\ + \beta(y_1, \dots, y_{n-1}; a, b; x) f(y_1, \dots, y_{n-1}, b), \quad x \in I. \end{aligned} \tag{1}$$

The latter three properties are crucial because they permit the development of all the classical geometric design algorithms in $\mathbb{E}$ (*e.g.*, de Casteljau algorithms), and they guarantee shape preserving proprties.

Example. Let us recall a simple procedure to build QEC-spaces on I [8]. Take

- any *sequence of weight functions on I*, *i.e.*, any sequence $(w_0, \dots, w_{n-1})$ such that, for $0 \leqslant i \leqslant n-1$, w_i is positive and C^{n-1-i} on I;
- any two-dimensional space $\mathbb{U} \subset C^0(I)$ supposed to be a *Chebyshev space on I* (for short, C-space on I: any non-zero element in $\mathbb{U}$ vanishes at most once in I, not counting possible multiplicities) containing constants, that is, any space spanned by $\mathbb{1}, U$ where U is strictly monotone on I.

As is classical one can define differential operators $L_0, \dots, L_{n-1}$ on $C^{n-1}(I)$ as follows:

$$L_0 F := \frac{F}{w_0}\ , \quad L_i F := \frac{1}{w_i} D L_{i-1} F, \quad 1 \leqslant i \leqslant n-1. \tag{2}$$

Then the set of all functions $F \in C^{n-1}(I)$ such that $L_{n-1}F \in \mathbb{U}$ is an $(n+1)$-dimensional QEC-space on I, denoted $QEC(w_0, \dots, w_{n-1}; \mathbb{U})$. For instance, given any positive numbers $p, q > n-1$, the linear space $\mathbb{E}_n^{p,q}$ spanned by the functions $1, x, \dots, x^{n-2}, (1-x)^p, x^q$ is a QEC-space on $[0,1]$, see [2], [3], [5].

We recently established the following converse property, of which the proof strongly relies on Theorem 2.1 and on the pseudoaffinity property of blossoms.

Theorem 2.2 *Let $\mathbb{E}$ be an $(n+1)$-dimensional QEC-space on a* **closed bounded interval** *I. Then there exists infinitely many ways to write $\mathbb{E}$ as $\mathbb{E} = QEC(w_0, \dots, w_{n-1}; \mathbb{U})$.*

3. The result

Given $I = [a, b]$, $a < b$, the ingredients to build our spline space $\mathbb{S}$ are :

- a sequence of interior knots: $a < t_1 < \cdots < t_q < b$ and an associated sequence of multiplicities m_k, with $1 \leqslant m_k \leqslant n$ for $1 \leqslant k \leqslant q$;
- a sequence of section spaces $\mathbb{E}_k$, $0 \leqslant k \leqslant q$: for each k, $\mathbb{E}_k$ contains the constant function $\mathbb{1}_k$ and $D\mathbb{E}_k$ is an n-dimensional QEC-space on $[t_k, t_{k+1}]$, where $t_0 := a$, $t_{q+1} := b$;

– a sequence of connection matrices M_k, $1 \leqslant k \leqslant q$: for each k, M_k is a lower triangular square matrix of order $(n - m_k)$ with positive diagonal entries.

The *spline space* $\mathbb{S}$ is then defined as the space of all continuous functions $S : I \to \mathbb{R}$ such that

(1) for $k = 0, \ldots, q$, the restriction of S to $[t_k, t_{k+1}]$ belongs to $\mathbb{E}_k$;

(2) for $k = 1, \ldots, q$, the following connection condition is fulfilled:

$$\big(S'(t_k^+), \ldots, S^{(n-m_k)}(t_k^+)\big)^T = M_k \cdot \big(S'(t_k^-), \ldots, S^{(n-m_k)}(t_k^-)\big)^T. \tag{3}$$

As a consequence of Theorems 2.1 and 2.2, for each $k = 0, \ldots, q$ one can choose a system $(w_1^k, \ldots, w_{n-1}^k)$ of weight functions on $[t_k, t_{k+1}]$ and a two-dimensional C-space $\mathbb{U}_k$ on $[t_k, t_{k+1}]$ containing the constant function $\mathbb{1}_k$ such that $\mathbb{E}_k = QEC(\mathbb{1}_k, w_1^k, \ldots, w_{n-1}^k; \mathbb{U}_k)$. For each such choice, when denoting by $L_0^k = \mathrm{Id}$, $L_1^k, \ldots, L_{n-1}^k$ the differential operators on $C^{n-1}([t_k, t_{k+1}])$ associated with $(\mathbb{1}_k, w_1^k, \ldots, w_{n-1}^k)$, the connection conditions (3) can be replaced by

$$\big(L_1^k(t_k^+), \ldots, L_{n-m_k}^k(t_k^+)\big)^T = \widehat{M}_k \cdot \big(L_1^{k-1}(t_k^-), \ldots, L_{n-m_k}^{k-1}(t_k^-)\big)^T, \quad 1 \leqslant k \leqslant q, \tag{4}$$

where $\widehat{M}_k$ is in turn a lower triangular matrix of order $(n - m_k)$ with positive diagonal entries.

The result we announce is then the following:

Theorem 3.1 *The following two statements are equivalent:*

(i) *the spline space $\mathbb{S}$ is good for design, in the sense that it possesses blossoms;*

(ii) *for each $k = 0, \ldots, q$, there exists a system $(w_1^k, \ldots, w_{n-1}^k)$ of weight functions on $[t_k, t_{k+1}]$ and a two-dimensional C-space $\mathbb{U}_k$ on $[t_k, t_{k+1}]$ containing the constant function $\mathbb{1}_k$ such that $\mathbb{E}_k = QEC(\mathbb{1}_k, w_1^k, \ldots, w_{n-1}^k; \mathbb{U}_k)$ and such that each matrix $\widehat{M}_k$ in (4) is the identity matrix of order $(n - m_k)$.*

Again, for the sake of simplicity we omit the precise definition of blossoms for splines, simply mentioning that, when (i) holds, each spline $S \in \mathbb{S}$ *"blossoms"* into a function s (the blossom of S), of which the natural domain of definition is a restricted set $\mathbb{A}_n(\mathbb{K})$ of n-tuples said to be *admissible* w.r.t. the knot-vector $\mathbb{K} := (\xi_{-n}, \ldots, \xi_{m+n+1}) := ({t_0}^{[n+1]}, {t_1}^{[m_1]}, \ldots, {t_q}^{[m_q]}, {t_{q+1}}^{[n+1]})$, with $m := \sum_{k=1}^q m_k$.

To conclude the section let us stress that Theorem 3.1 presents a twofold interest:

- first, it provides us with a very simple recipe to build all good spline spaces;
- for given section spaces, it enables us to answer the following question: a sequence $\big(M_k\big)_{k=1}^q$ of connection matrices being given, is the corresponding spline space suitable for design or not?

4. Sketch of the proof – Illustration

Theorem 3.1 extends to QEC-spaces a recent result obtained for EC-spaces [10]. Though guided by the ideas developed in [10], replacing EC-spaces by QEC-spaces is not a trivial adaptation of existing results, especially as we are working with different section spaces. Below we mention the two majors difficulties encountered in the present situation:

• When (i) holds, blossoms are obviously symmetric on $\mathbb{A}_n(\mathbb{K})$ and they obviously satisfy the diagonal property. A major difficulty consists in proving that they also satisfy the crucial pseudoaffinity property $(\mathrm{B})_3$, of course limited to $\mathbb{A}_n(\mathbb{K})$. The tricky proof of $(\mathrm{B})_3$ in QEC-spaces [8] involved difficult generalised convexity arguments. The spline framework will require that we go deeper into the arguments in question. We would like to stress that pseudoaffinity is THE property justifying that a spline space is considered good for design when it possesses blossoms. Indeed, it permits all the geometric design algorithms and leads to the important intermediate result stated below (see [6] for the EC case):

Theorem 4.1 *If* (i) *holds, then* $\mathbb{S}$ *possesses a quasi B-spline basis (see below) which is its optimal normalised totally positive basis. Conversely, if* $\mathbb{S}$ *and any spline space derived from* $\mathbb{S}$ *by knot insertion possess B-spline bases, then* (i) *holds.*

Normalised totally positive bases are crucial for design [4], and optimality should simply be understood as "the best possible" [1]. For the links with blossoms, see [7]. The presence of *quasi B-spline basis* is crucial too. Let us recall that it means a normalised sequence $N_\ell \in \mathbb{S}$, $-n \leqslant \ell \leqslant m$, (*i.e.*, $\sum_{\ell=-n}^{m} N_\ell = \mathbb{1}$), each N_ℓ being positive on the interior of its support $[\xi_\ell, \xi_{\ell+n+1}]$, and satisfying some additional condition of zeros at the endpoints of its support. The term "quasi" simply refers to the fact that we are dealing with QEC-spaces, not with EC-spaces, and the count of zeros should take this into account.

• For simplicity we have assumed all interior mulitplicities to satisfy $m_k \geqslant 1$. Still, it is essential to understand what does $m_k = 0$ would mean when dealing with QEC-space (see Example below).

Example. Let us give an elementary example. Given any real numbers $p_k, q_k > n-1$, $0 \leqslant k \leqslant q$, here we consider a spline space $\mathbb{S}$ with kth section space $\mathbb{E}_n^{p_k,q_k}$ up to the convenient affine change of variable.

1- To remain inside the framework described at the beginning of Section 3, it is sufficient to consider the case where all interior knots are simple. Given any positive numbers a_k, $1 \leqslant k \leqslant q$, the splines are then supposed to be C^{n-2} on I and to meet the additional requirement:

$$S^{(n-1)}(t_k^+) = a_k S^{(n-1)}(t_k^-), \quad 1 \leqslant k \leqslant q. \tag{5}$$

Theorem 3.1 guarantees that the spline space $\mathbb{S}$ is then good for design. In particular it therefore possesses a quasi B-spline basis which is its optimal normalised totally positive basis.

2- To illustrate multiplicities equal to 0, let us now assume that $m_k \in \{0,1\}$ for all $k = 1, \ldots, q$. Any interior knot t_k of multiplicity $m_k = 0$ is then allocated two additional positive numbers b_k, c_k. Along with (5) the splines in $\mathbb{S}$ are now supposed to satisfy:

$$S^{(n-1)}(t_{k+1}^-) = -b_k S^{(n-1)}(t_{k-1}^+) + c_k S^{(n-1)}(t_k^-) \text{ for any } k \in \{1, \ldots, q\} \text{ such that } m_k = 0. \tag{6}$$

Again, this yields a spline space good for design.

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